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ERROR BOUNDS FOR THE LIOUVILLE-GREEN APPROXIMATION TO INITIAL-V--ETC(U)
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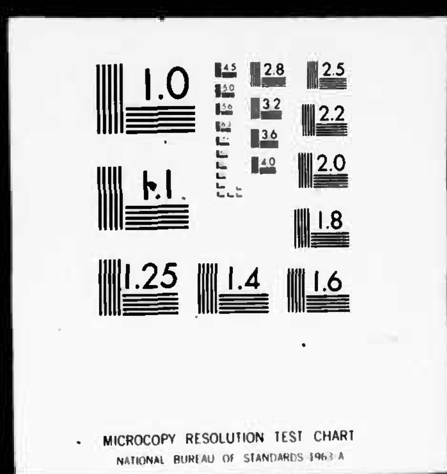

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Error Bounds for the Liouville-Green Approximation to Initial-Value Problems

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

New error bounds are developed for the Liouville-Green approximation to the solution of an important class of differential equations arising in military operations research (specifically, variable-coefficient Lanchester-type equations of modern warfare for combat between two homogeneous forces). In contrast to many previous results, our error bounds apply to initial-value

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problems and are expressed in terms of initial conditions. Previous error bounds for boundary-value problems are sharpened as a consequence of our development of these new error bounds for initial-value problems. Finally, applications are made to some important specific models of combat between two homogeneous forces with time-dependent attrition-rate coefficients.

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SUMMARY

New error bounds are developed for the Liouville-Green approximation to the solution of an important class of differential equations arising in military operations research (specifically, variable-coefficient Lanchester-type equations of modern warfare for combat between two homogeneous forces). In contrast to many previous results, our error bounds apply to initial-value problems and are expressed in terms of initial conditions. Previous error bounds for boundary-value problems are sharpened as a consequence of our development of these new error bounds for initial-value problems. Finally, applications are made to some important specific models of combat between two homogeneous forces with time-dependent attrition-rate coefficients.

1. Introduction

OLVER [12] has developed error bounds for the so-called LIOUVILLE-GREEN[1)] (LG) approximation [5, 10] to the solution of the differential equation

$$\frac{d^2x}{dt^2} = J(t)x. \tag{1}$$

His results extended earlier work by BLUMENTHAL [1]. The LG approximation is of particular importance in applied mathematics because it involves only elementary functions. As OLVER [12] has pointed out, however, the development of strict upper bounds for the errors in the approximate solutions has received relatively little attention. Moreover, the error bounds that have been previously developed [12, 13] are for boundary-value problems, since most problems of interest in mathematical physics are boundary-value problems. These error bounds do not apply to initial-value problems.

Thus, the purpose of this paper is to develop error bounds for the LG approximation to initial-value problems. Furthermore, in recasting OLVER's results in a form suitable for initial-value problems, we have been able to sharpen his bounds for boundary-value problems. Although we will develop our results within the context of a specific problem in operations research, they are clearly applicable to the general second order initial-value problem, and it is a straightforward task to so recast them.

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1) Also called the WKB approximation [13].

Probably the most widely used (at least in the United States) deterministic differential-equation model in operations research [4,15-17] are LANCHESTER-type equations of warfare[2], which are named for the pioneering 1914 work of F. W. LANCHESTER [9]. In this paper we consider such a linear, variable-coefficient differential-equation model for combat between two homogeneous forces. This model yields an X force-level equation equivalent to (1). Unfortunately, for even the simplest time-varying attrition-rate coefficients of interest, the solutions cannot in general be expressed in terms of either elementary functions or tabulated higher transcendental functions [17]. It is therefore natural to seek an approximation in terms of elementary functions. Moreover, one is always interested in a simple a priori estimate for the error in the approximate solution. In this paper we develop an error bound that is both realistic and easy to evaluate. This bound is then computed for some particular attrition-rate coefficients of interest.

2. Variable-Coefficient LANCHESTER-Type Equations of Modern Warfare

We consider the following variable-coefficient LANCHESTER-type equations of modern warfare[3] for combat between two homogeneous forces

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[2] We will refer to any differential-equation model of combat as being LANCHESTER-type equations. The state variables are typically the force levels of the various weapon system types.

[3] The term "of modern warfare" denotes that we are considering linear differential equations. There are other nonlinear types of LANCHESTER equations [4,16].

$$dx/dt = -a(t)y, \qquad dy/dt = -b(t)x, \tag{2}$$

with initial conditions

$$x(t=0) = x_0, \qquad y(t=0) = y_0,$$

where $t=0$ denotes the time at which the battle begins, $x(t)$ and $y(t)$ denote the numbers of X and Y at time t, and $a(t)$ and $b(t)$ denote time-dependent LANCHESTER attrition-rate coefficients. These equations (2) have been hypothesized to model combat in which both sides use aimed fire and target acquisition times are independent of the numbers of firers and targets [2,19]. The attrition-rate coefficients represent the effectiveness of each side's fire (i.e. its firepower). Temporal variations in a side's fire effectiveness (caused by, for example, changes in force separation, combatant postures, target acquisition rates, etc.) are modelled by the time-dependent attrition-rate coefficients. Further discussions of the physical assumptions hypothesized to yield (2), estimation of the attrition-rate coefficients, and the importance of (2) to military operations research are found in [16,17].

We assume that $a(t)$ and $b(t)$ are positive and twice differentiable for $t_0 < t < +\infty$ with $t_0 < 0$. We also assume that $a(t), b(t) \in L(t_0,T)$ for any finite T. We further take $a(t)$ and $b(t)$ to be given in the form $a(t) = k_a g(t)$, $b(t) = k_b h(t)$, where k_a, k_b are positive constants chosen so that $a(t)/b(t) = k_a/k_b$ when $g(t) = h(t)$ for all t. We introduce (see [17]) the <u>intensity of combat</u> $I(t)$ and the <u>relative fire effectiveness</u> $R(t)$ defined by

$$I(t) = \sqrt{a(t)b(t)}, \qquad \text{and} \qquad R(t) = a(t)/b(t). \tag{3}$$

We similarly introduce the <u>combat-intensity parameter</u> λ_I and the <u>relative-fire-effectiveness parameter</u> λ_R defined by

$$\lambda_I = \sqrt{k_a k_b}, \qquad \text{and} \qquad \lambda_R = k_a/k_b. \tag{4}$$

A large class of tactical situations of interest can be modelled with the following general power attrition-rate coefficients [17]

$$a(t) = k_a(t+C)^{\mu}, \qquad \text{and} \qquad b(t) = k_b(t+C+A)^{\nu}, \tag{5}$$

where $A,C \geq 0$. We will call A the <u>offset parameter</u>, since it allows us to model (with $\mu,\nu \geq 0$) battles between weapon systems with different maximum effective ranges. We will call C the <u>starting parameter</u>, since it allows us to model (again, with $\mu,\nu \geq 0$) battles that begin within the minimum of the maximum effective ranges of the two systems. The offset and starting parameters are related to various physical quantities in [17]. We observe that $t_0 = -C$. Also, $a(t), b(t) \in L(t_0,T)$ implies that $\mu,\nu > -1$.

From (2) we may obtain the X force-level equation

$$\frac{d^2x}{dt^2} - \left\{\frac{1}{a(t)}\frac{da}{dt}\right\}\frac{dx}{dt} - a(t)b(t)x = 0, \tag{6}$$

with initial conditions

$$x(t=0) = x_0, \qquad \text{and} \qquad \{[1/a(t)]dx/dt\}_{t=0} = -y_0.$$

We may consider that $t_0 = \max(t_0^X, t_0^Y)$, where t_0^X denotes the right-most finite singularity of the X force-level equation. Furthermore, we set $t_0 = 0$ if there are no finite singularities.

3. LIOUVILLE-GREEN Approximation to LANCHESTER-Type Equations of Modern Warfare

Let

$$\tau = \int_{t_0}^{t} \sqrt{a(s)b(s)}\, ds = \int_{t_0}^{t} I(s)ds, \tag{7}$$

and denote $\tau(t=0)$ as τ_0. Then $\tau_0 \geq 0$ for $t_0 \leq 0$. From $a(t), b(t) \in L(t_0,T)$ it follows that $\tau = \tau(t)$ is well defined by the CAUCHY-SCHWARZ inequality for integrals. The transformation is also easily seen to be invertable. We observe that $\tau - \tau_0$ is related to the average intensity of combat $\bar{I}(t)$ by

$$\tau - \tau_0 = \left\{\frac{1}{t}\int_{0}^{t} I(s)ds\right\} t = t\bar{I}(t). \tag{8}$$

We may call $\tau - \tau_0$ "the elapsed normalized battle time," since the transformation (7) reparameterizes the battle's time scale in terms of elapsed time of battle and average combat intensity as (8) shows us. The substitution (7) transforms (6) into

$$\frac{d^2x}{d\tau^2} - \left\{\frac{1}{2}\frac{d}{d\tau}\ln R(t)\right\}\frac{dx}{d\tau} - x = 0, \tag{9}$$

with initial conditions

$$x(\tau=\tau_0) = x_0, \quad \text{and} \quad \{[1/R^{1/2}(t)]dx/d\tau\}_{\tau=\tau_0} = -y_0.$$

Remark 1: It is easily shown (see [17]) that (9) may be transformed into a linear second order differential equation with constant coefficients if and only if

$$\frac{1}{I(t)}\frac{d}{dt}\ln R(t) = \text{constant}.$$

Let R_0 denote $R(t=0)$. The substitution

$$x(\tau) = X(\tau)[R(t)/R_0]^{1/4}, \tag{10}$$

transforms (9) into LIOUVILLE's normal form (see INCE [7] or KAMKE [8])

$$\frac{d^2X}{d\tau^2} - \{1+F(\tau)\}X = 0, \tag{11}$$

with initial conditions

$$X(\tau=\tau_0) = x_0, \text{ and } dX/d\tau(\tau=\tau_0) = -y_0\sqrt{R_0} - x_0\varepsilon_0,$$

where

$$F(\tau) = P''(\tau)/P(\tau), \qquad P(\tau) = [R(t)]^{-1/4}, \tag{12}$$

$$\varepsilon(t) = \frac{1}{4I(t)} \frac{d}{dt} \ln R, \tag{13}$$

ε_0 denotes $\varepsilon(t=0)$, and $P'(\tau)$ denotes $dP/d\tau$.

Writing (11) as $d^2X/d\tau^2 - X = F(\tau)X$, we may use variation of parameters to obtain the solution to (11) as

$$X(\tau) = x_0 \cosh(\tau-\tau_0) - (y_0\sqrt{R_0}+x_0\varepsilon_0)\sinh(\tau-\tau_0) + \int_{\tau_0}^{\tau} F(\sigma)\sinh(\tau-\sigma)X(\sigma)d\sigma. \tag{14}$$

If one drops[4)] the integral term in (14), one obtains the LIOUVILLE-GREEN approximation $\hat{X}(\tau)$

[4)] Heuristically, if the appropriate fractional power of the relative effectiveness $R(t)$ is "slowly varying," then by (12) we would expect that $|F(\tau)| \ll 1$ so that the integral term in (14) is "negligible." Theorem 1 gives us bounds on how "negligible" this term is.

$$\hat{x}(\tau) = x_0 \cosh(\tau-\tau_0) - (y_0\sqrt{R_0}+x_0\varepsilon_0)\sinh(\tau-\tau_0), \tag{15}$$

which in terms of the original independent variable x reads

$$\hat{x}(t) = \left[\frac{R(t)}{R_0}\right]^{1/4} \{x_0 \cosh(\tau-\tau_0) - (y_0\sqrt{R_0}+x_0\varepsilon_0)\sinh(\tau-\tau_0)\}. \tag{16}$$

We observe from (14) that $F(\tau) \geq 0$ for all $\tau \geq \tau_0$ implies that as long as $x(t) \geq 0$ we have $x(t) \geq \hat{x}(t)$. A similar statement holds for $F(\tau) \leq 0$. As we shall see below, such cases in which $F(\tau)$ is always ≥ 0 or ≤ 0 are readily encountered in applications.

4. <u>Error Bounds for the LIOUVILLE-GREEN Approximation</u>

The main result of this paper is Theorem 1.

Theorem 1: <u>Error bounds for the LIOUVILLE-GREEN approximation are given by</u>

$$|x(t)-\hat{x}(t)| \leq x_0 K_J e(t) < x_0 K_U e(t), \tag{17}$$

<u>where</u>

$$K_U = 2\{(1+|\varepsilon_0|) + \frac{y_0}{x_0}\sqrt{R_0}\}, \tag{18}$$

$$J = I \quad \text{for} \quad 1-\frac{y_0}{x_0}\sqrt{R_0} \leq \varepsilon_0 \quad \text{and} \quad K_I = 1+\varepsilon_0+\frac{y_0}{x_0}\sqrt{R_0}, \tag{19}$$

$$J = II \quad \text{for} \quad -1-\frac{y_0}{x_0}\sqrt{R_0} < \varepsilon_0 < 1-\frac{y_0}{x_0}\sqrt{R_0} \quad \text{and} \quad K_{II} = 2, \tag{20}$$

$$J = III \quad \text{for} \quad \varepsilon_0 \leq -1-\frac{y_0}{x_0}\sqrt{R_0} \quad \text{and} \quad K_{III} = 1-\varepsilon_0-\frac{y_0}{x_0}\sqrt{R_0} > 0, \tag{21}$$

<u>and</u>

$$e(t) = \left[\frac{R(t)}{R_0}\right]^{1/4} \{\exp(\frac{1}{2}\int_{\tau_0}^{\tau} |F(\sigma)|d\sigma)-1\}\sinh(\tau-\tau_0). \qquad (22)$$

The sign of the error is determined by the sign of $F(\tau)$. As long as $x(t) \geq 0$, it follows that

$F(\tau) \geq 0$ for all $\tau \geq 0$ implies that $x(t) \geq \hat{x}(t)$,

with the last inequality being reversed when $F(\tau) \leq 0$.

Proof: Theorem 1 readily follows from Lemma 2, which is proven below.

5. Development of Error Bounds

Consider the following fundamental system of solutions $\{X_1, X_2\}$ to (11)

$$X_k(\tau) = \{1+h_k(\tau)\}\exp[(-1)^{(k-1)}(\tau-\tau_0)] \quad \text{for} \quad k = 1,2, \qquad (23)$$

where $h_k(\tau)$ for $k = 1,2$ is to be chosen so that

$$h_k(\tau=\tau_0) = 0, \quad \text{and} \quad dh_k/d\tau(\tau=\tau_0) = 0. \qquad (24)$$

It follows that the solution to (11) may be expressed as

$$X(\tau) = \frac{1}{2}\{(x_0(1-\varepsilon_0)-y_0\sqrt{R_0})e^{(\tau-\tau_0)}[1+h_1(\tau)] + (x_0(1+\varepsilon_0)+y_0\sqrt{R_0})e^{-(\tau-\tau_0)}[1+h_2(\tau)]\}, \qquad (25)$$

so that

$$X(\tau) - \hat{X}(\tau) = \frac{1}{2}\{[x_0(1-\varepsilon_0)-y_0\sqrt{R_0}]e^{(\tau-\tau_0)}h_1(\tau) + [x_0(1+\varepsilon_0)+y_0\sqrt{R_0}]e^{-(\tau-\tau_0)}h_2(\tau)\}. \qquad (26)$$

Substituting (23) into (11), we find that $h_k(\tau)$ for $k = 1,2$ must satisfy

$$\frac{d^2h_k}{d\tau^2} - (-1)^k\, 2\frac{dh_k}{d\tau} - F(\tau)h_k = F(\tau), \tag{27}$$

with initial conditions (24).

We may consider $h_k(\tau)$ for $k = 1,2$ to be an error term for the LIOUVILLE-GREEN approximation. We next develop a bound on its magnitude, which sharpens earlier results by OLVER [12].

Lemma 1: A bound on the magnitude of $h_k(\tau)$ for $k = 1,2$ is given by

$$|h_k(\tau)|\exp[(-1)^{(k-1)}(\tau-\tau_0)] \le 2\{\exp(\frac{1}{2}\int_{\tau_0}^{\tau}|F(\sigma)|d\sigma)-1\}\sinh(\tau-\tau_0). \tag{28}$$

Proof: For notational convenience, we develop the bounds for $h_1(\tau)$ and $h_2(\tau)$ separately. Transposing the right-most term on the left-hand side of (27) for $k = 1$, treating $\{1+h_1(\tau)\}F(\tau)$ as a "forcing term," and integrating twice; we obtain the following VOLTERRA integral equation after a further integration by parts

$$h_1(\tau) = \frac{1}{2}\int_{\tau_0}^{\tau}\{1-e^{2(\sigma-\tau)}\}F(\sigma)\{1+h_1(\sigma)\}d\sigma. \tag{29}$$

Solving (29) in the usual manner by successive approximations, we obtain

$$h_1(\tau) = \sum_{n=1}^{\infty} T_n(\tau), \tag{30}$$

where $T_0(\tau) = 1$ and for $n \ge 1$

$$T_n(\tau) = \frac{1}{2}\int_{\tau_0}^{\tau}\{1-e^{2(\sigma-\tau)}\}F(\sigma)T_{n-1}(\sigma)d\sigma. \tag{31}$$

Observing that $\{1-e^{2(\sigma-\tau)}\} \le 1-e^{-2(\tau-\tau_0)}$ for $0<\tau_0 \le \sigma \le \tau$, we find that

$$|T_1(\tau)| \le \frac{\left\{1-e^{-2(\tau-\tau_0)}\right\}}{2} \int_{\tau_0}^{\tau} |F(\sigma)| d\sigma ,$$

with equality holding for $\tau = \tau_0$. A straightforward inductive argument along the usual lines now yields

$$|T_n(\tau)| \le \frac{\left\{1-e^{-2(\tau-\tau_0)}\right\}}{2^n\, n!} \left\{\int_{\tau_0}^{\tau} |F(\sigma)| d\sigma\right\}^n . \tag{32}$$

One step in the inductive proof of (32) is deserving of further elaboration, however. From (31) and (32) we obtain

$$|T_{n+1}(\tau)| \le \frac{1}{2^{n+1}\, n!} \int_{\tau_0}^{\tau} \{1-e^{2(\sigma-\tau)}\}\left\{1-e^{-2(\sigma-\tau_0)}\right\} |F(\sigma)| \left\{\int_{\tau_0}^{\sigma} |F(u)| du\right\}^n d\sigma . \tag{33}$$

The inductive proof of (32) is completed by combining (33) with the observation that $\{1-e^{2(\sigma-\tau)}\}\left\{1-e^{-2(\sigma-\tau_0)}\right\} \le 1-e^{-2(\tau-\tau_0)}$ for $0<\tau_0 \le \sigma \le \tau$. Our sharpening of OLVER's results is due to this observation. The remaining steps in the proof of (28) for $k = 1$ follow along well-known lines [12,13] and will be omitted here. Similar arguments are used to prove (28) for $k = 2$.

Remark 2: In our notation, OLVER's [12] corresponding error bound for $k=1$ would read

$$|h_1(\tau)| \le \exp\left(\frac{1}{2}\int_{\tau_0}^{\tau} |F(\sigma)| d\sigma\right) - 1 . \tag{34}$$

His corresponding error bound for $k=2$ is not directly comparable to our result here, since he does not take both errors zero at the same point.[5)]

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5) In our notation for the finite or infinite interval (τ_0, τ_1), OLVER [12,13] takes $h_1(\tau=\tau_0) = dh_1/d\tau(\tau=\tau_0) = 0$ and $h_2(\tau=\tau_1) = dh_2/d\tau(\tau=\tau_1) = 0$ in contrast to (24). This is what makes his results unsuitable for initial-value problems.

Remark 3: Similar arguments may be used to develop a bound on $|h_k'(\tau)|$. For our applications this result is not important.

Lemma 2: In terms of the transformed dependent variable $X(\tau)$, error bounds for the LIOUVILLE-GREEN approximation are given by

$$|X(\tau)-\hat{X}(\tau)| \le x_0 K_J E(\tau) < x_0 K_U E(\tau), \tag{35}$$

where K_U and K_J are given by (18) through (21), and

$$E(\tau) = \{\exp(\frac{1}{2}\int_{\tau_0}^{\tau} |F(\sigma)| d\sigma) - 1\}\sinh(\tau-\tau_0). \tag{36}$$

Proof: From (26) and Lemma 1 we obtain

$$|X(\tau)-\hat{X}(\tau)| \le \{|x_0(1-\varepsilon_0)-y_0\sqrt{R_0}| + |x_0(1+\varepsilon_0)+y_0\sqrt{R_0}|\}E(\tau). \tag{37}$$

It follows that a rather loose error bound is given by

$$|X(\tau)-\hat{X}(\tau)| \le x_0 K_U E(\tau). \tag{38}$$

We observe that for $-1-\frac{y_0}{x_0}\sqrt{R_0} < \varepsilon_0 < 1-\frac{y_0}{x_0}\sqrt{R_0}$, we have

$$|x_0(1-\varepsilon_0)-y_0\sqrt{R_0}| + |x_0(1+\varepsilon_0)+y_0\sqrt{R_0}| = 2x_0, \tag{39}$$

so that (37) becomes $|X(\tau)-\hat{X}(\tau)| \le 2x_0 E(\tau)$. Thus, (35) is proven for $J = II$. For $1-\frac{y_0}{x_0}\sqrt{R_0} \le \varepsilon_0$, the error bound (37) becomes

$$|X(\tau)-\hat{X}(\tau)| \le 2x_0(\varepsilon_0 + \frac{y_0}{x_0}\sqrt{R_0})E(\tau), \tag{40}$$

since

$$|x_0(1-\varepsilon_0)-y_0\sqrt{R_0}| + |x_0(1+\varepsilon_0)+y_0\sqrt{R_0}| = 2x_0(\varepsilon_0 + \frac{y_0}{x_0}\sqrt{R_0}). \tag{41}$$

The error bound (40) may be sharpened, however, as follows. If $F(\sigma) \geq 0$ for $0 < \tau_0 \leq \sigma \leq \tau$, then as long as $X(\tau) \geq 0$ we have $X(\tau) \geq \hat{X}(\tau)$ and $g_k(\tau) \geq 0$, where for $k = 1,2$

$$g_k(\tau) = \frac{1}{2} h_k(\tau) \exp[(-1)^k(\tau-\tau_0)]. \tag{42}$$

It follows from (26) that for $1 - \frac{y_0}{x_0}\sqrt{R_0} \leq \varepsilon_0$

$$0 \leq X(\tau) - \hat{X}(\tau) \leq \{x_0(1+\varepsilon_0) + y_0\sqrt{R_0}\}E_+(\tau), \tag{43}$$

where

$$E_+(\tau) = \{\exp(\frac{1}{2}\int_{\tau_0}^{\tau} F(\sigma)d\sigma) - 1\}\sinh(\tau-\tau_0). \tag{44}$$

Similarly, if $F(\sigma) \leq 0$ for $\tau_0 \leq \sigma \leq \tau$, then as long as $X(\tau) \geq 0$ we have $X(\tau) \leq \hat{X}(\tau)$ and $g_k(\tau) \leq 0$, so that

$$0 \geq X(\tau) - \hat{X}(\tau) \geq \{x_0(1+\varepsilon_0)+y_0\sqrt{R_0}\}E_-(\tau), \tag{45}$$

where

$$E_-(\tau) = \{1-\exp(-\frac{1}{2}\int_{\tau_0}^{\tau} F(\sigma)d\sigma)\}\sinh(\tau-\tau_0). \tag{46}$$

It follows from (43) and (45) that

$$|X(\tau) - \hat{X}(\tau)| \leq x_0\{1 + \varepsilon_0 + \frac{y_0}{x_0}\sqrt{R_0}\}E(\tau). \tag{47}$$

Furthermore, since $\varepsilon_0 \geq 1 - \frac{y_0}{x_0}\sqrt{R_0}$ implies that $2(\varepsilon_0 + \frac{y_0}{x_0}\sqrt{R_0}) \geq 1 + \varepsilon_0 + \frac{y_0}{x_0}\sqrt{R_0}$, the bound given by (47) is sharper than that given by (40). Thus, (35) is proven for $J = I$. The proof of (35) for $J = III$ is similar to that for $J = II$. It is easily seen that $K_J < K_U$ for $x_0, y_0 > 0$.

6. Examples

We now compute theoretical error bounds for two special cases of the LIOUVILLE-GREEN approximation to the solution of (6) with the general power attrition-rate coefficients (5): (I) power attrition-rate coefficients with no offset (i.e. $A=0$), and (II) linear attrition-rate coefficients with positive offset (i.e. $A>0$).

6.1. Power Attrition-Rate Coefficients with No Offset

In this case we have

$$a(t) = k_a(t+C)^{\mu}, \quad \text{and} \quad b(t) = k_b(t+C)^{\nu}, \tag{48}$$

with $C>0$ and $\mu,\nu>-1$. The LIOUVILLE-GREEN approximation to the X force level is

$$\begin{aligned}\hat{x}(t) = (1+t/C)^{(\mu-\nu)/4}\{&x_0 \cosh(\tau-\tau_0) \\ &-[y_0\sqrt{\lambda_R}\, C^{(\mu-\nu)/2} + \frac{x_0(\mu-\nu)}{4\lambda_I} C^{-\delta}]\sinh(\tau-\tau_0)\},\end{aligned} \tag{49}$$

where

$$\tau(t) = (1/\delta)\lambda_I(t+C)^{\delta}, \tag{50}$$

and

$$\delta = (\mu+\nu+2)/2. \tag{51}$$

In preparation for estimating the error in the LIOUVILLE-GREEN approximation (49) by Theorem 1, we compute

$$F(\tau) = \frac{(\mu-\nu)(3\mu+\nu+4)}{16\delta^2\tau^2}. \tag{52}$$

Thus, $F(\tau) \geq 0$ for all $\tau \geq \tau_0 > 0$ if and only if $\mu \geq \nu$. Hence, as noted at the end of Section 3, one frequently encounters in applications cases in which $F(\tau)$ always has the same sign. For the error estimate (17), we then have

$$\frac{1}{2}\int_{\tau_0}^{\tau} |F(\sigma)|d\sigma = \frac{|\mu-\nu|(3\mu+\nu+4)}{32\lambda_I\delta}\{C^{-\delta}-(t+C)^{-\delta}\}. \tag{53}$$

Remark 4: The exact solution $x(t)$ to (6) with attrition-rate coefficients (48) is given in TAYLOR and BROWN [17] (see also [16]). It may be expressed in terms of modified Bessel functions of the first kind of (for $\mu,\nu > -1$) fractional order, i.e. I_α for $0 < \alpha < 1$. Since few of the latter are tabulated (i.e. tabulations only exist for $\alpha = \pm 1/4, \pm 1/3, \pm 1/2, \pm 2/3, \pm 3/4$, and these do correspond to cases of interest), TAYLOR and BROWN [17] suggested the use of new transcendents which they called LANCHESTER-CLIFFORD-SCHLÄFLI functions[6].

6.2. Linear Attrition-Rate Coefficients with Positive Offset

$$a(t) = k_a(t+C), \qquad \text{and} \qquad b(t) = k_b(t+C+A), \tag{54}$$

with $A,C > 0$. The LIOUVILLE-GREEN approximation to the X force level is

[6] After earlier work by W. K. CLIFFORD [3] and L. SCHLÄFLI [14] (see also [6,18]).

$$\hat{x}(t) = \left[\frac{(1+A/C)}{(1+A/(t+C))}\right]^{1/4} \{x_0 \cosh(\tau-\tau_0)$$

$$- \left[\frac{y_0\sqrt{\lambda_R}}{\sqrt{1+A/C}} + \frac{x_0 A/C}{4\lambda_I C^2 (1+A/C)^{3/2}}\right] \sinh(\tau-\tau_0)\}, \qquad (55)$$

where

$$\tau(t) = \frac{A^2}{8}\lambda_I\{\psi\sqrt{\psi^2-1} - \ln(\psi+\sqrt{\psi^2-1})\}, \qquad (56)$$

and

$$\psi(t) = 1 + 2(t+C)/A. \qquad (57)$$

For estimating the error in the LIOUVILLE-GREEN approximation (55), it is more convenient to express $F(\tau)$ as defined by (12) in terms of the original independent variable t. We compute that[7]

$$F(\tau) = \frac{A\{12(t+C) + 7A\}}{16\lambda_I^2 (t+C)^3 (t+C+A)^3}. \qquad (58)$$

It follows that $F(\tau) > 0$ for all $\tau \geq \tau_0 > 0$. Thus, as in the previous example, $F(\tau)$ always has the same sign. For the error estimate (17), we then have

$$0 \leq \frac{1}{2}\int_{\tau_0}^{\tau} F(\sigma)d\sigma \leq \min(m_1(t), m_2(t)), \qquad (59)$$

where

$$m_1(t) = \frac{3A}{16C^3(1+A/C)^{5/2}\lambda_I}\{4(1-q^{1/2}) + \frac{7A}{4C}(1-q^{3/2})\}, \qquad (60)$$

$$m_2(t) = \frac{A}{32C^3\lambda_I}\{4(1-q^3) + \frac{7A}{4C}(1-q^4)\}, \qquad (61)$$

---

[7] In general

$$F(\tau) = \frac{1}{4b^2(t)}\frac{d}{dt}\{-\frac{d}{dt}\ln b(t) - \frac{1}{4}\frac{d}{dt}\ln R(t) + \frac{d}{dt}\ln\frac{dR}{dt}\}.$$

and

$$q(t) = 1/(1+t/C). \tag{62}$$

It may be shown that $m_1(t_1) = m_2(t_1)$ implies that $m_1(t) > m_2(t)$ for all $t > t_1$. The error control term estimate $m_1(t)$ was developed for "small" t[8], while $m_2(t)$ for "large" t[9].

Remark 5: This case is of particular interest in military operations research, since it may be used to study combat between two weapon systems with different maximum effective ranges [16]. The exact solution $x(t)$ to (6) with attrition-rate coefficients (54) is given in [17] (see also [16]). It apparently cannot be expressed in terms of previously "known" transcendents, since the X force-level equation in this case could not be found to correspond to any second order linear equation considered in [8] or [11].

7. Final Remarks

Although given within the context of a specific problem in operations research, the reader will have no trouble translating the above results into those for the general second order

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8) Letting $J = \int_0^t \frac{\{12(s+C)+7A\}ds}{(s+C)^{5/2}(s+C+A)^{5/2}}$, we find that $\int_{\tau_0}^{\tau} F(\sigma)d\sigma = \frac{AJ}{16\lambda_I}$. Here we have used the bound $J \le \frac{1}{(C+A)^{5/2}} \int_0^t \frac{\{12(s+C)+7A\}ds}{(s+C)^{5/2}}$.

9) In this case (to be contrasted with the previous one), we have used the bound $J \le \int_0^t \frac{\{12(s+C)+7A\}ds}{(s+C)^5}$.

initial-value problem. In the examples of Section 6 we saw that for two models of considerable importance in military operations research both the LIOUVILLE-GREEN approximation and bounds on its error were simply expressed in terms of elementary functions. No previous application of the LIOUVILLE-GREEN approximation has appeared in the operations research literature. In a subsequent paper we plan to present a numerical investigation of the accuracy (both numerical eveluation of the theoretical error bounds and a comparison with the exact solution when available) of the LIOUVILLE-GREEN approximation to the solution of LANCHESTER-type equations of modern warfare for combat between two homogeneous forces.

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